Yiddish of Day

"A moshel iz nisht keyn raych" = אַ מאָשעל איז
נישט קײן רײַעה

"An example is not a proof" =

Subspaces

Last time

- Recall that for $X \subseteq \mathbb{F}^n$ we defined

$$
\begin{array}{lcl}
\mathrm{Fct}(X, \mathbb{F}) & = & \text{functions } f : X \to \mathbb{F} \\
\quad \cup\!| & & \\
\mathrm{Cts}(X, \mathbb{F}) & = & \text{cts functions } f : X \to \mathbb{F} \\
\quad \cup\!| & & \\
\mathrm{Diff}(X, \mathbb{F}) & = & \text{diff functions } f : X \to \mathbb{F}
\end{array}
$$

These are all subsets, but they have more structure,
They are themselves vectorspaces

Def: Let V be a $\mathbb{F}$-vs, $W \subseteq V$ subset,
We say W is a subspace if

1) $0_V \in W$

2) if $w_1, w_2 \in W$ then $w_1 + w_2 \in W$

3) $\forall c \in \mathbb{F}, w \in W \quad cw \in W$

- it will often be usefull to "break apart" the vector space V into smaller decomposition of subspaces. (we will return to this idea)

- Common occurance of subspaces

  - Def: Let V be an $\mathbb{F}$-vs and $(v_1, \ldots, v_k)$ in V. Then we say $w \in V$ is a linear-combo of these vectors if

$w = c_1 v_1 + \cdots + c_k v_k$ for some $c_1 \ldots c_k \in \mathbb{F}$

"generation"

<u>Def</u>: Let non-empty $S \subseteq V$ be a subset of V.

The <u>span</u> of S is the set

$$\langle S \rangle = \mathrm{span}(S) = \left\{ \sum_i c_i v_i \;\middle|\; \begin{array}{l} i \in \mathbb{N} \\ c_i \in \mathbb{F},\ v_i \in S \end{array} \right\}$$

= all possible linear combo's of vectors in S.

<u>Lemma</u>: <u>$\mathrm{Span}(S)$</u> is a subspace of V.

$s_1 \dots s_k \in S$
$\tilde{s}_1 \dots \tilde{s}_\ell$
$c_i, d_i \in \mathbb{F}$

Pf) Q: Is $0_V \in \text{Span}(S)$. Yes, take any $s \in S$.

Then $0_{\mathbb{F}} s = 0_V \in \text{Span}(S)$.

Take $v = c_1 s_1 + \dots + c_k s_k$, $w = d_1 \tilde{s}_1 + \dots + d_\ell \tilde{s}_\ell$

then $v + w = c_1 s_1 + \dots + c_k s_k + d_1 \tilde{s}_1 + \dots + d_\ell \tilde{s}_\ell \in \text{Span}(S)$

Similarly $\alpha v \in \text{Span}(S)$ $\forall \alpha \in \mathbb{F}$ ☺

HW: Show Span(S) is the smallest subspace containing S

- We often pay particular attention to how a vector is a linear-combo of a list of vectors.

Def: Say a list of vectors are linearly independent

if the only way 0_V is a LC of these vectors, is

if all the coefficients are 0_F

Why care? "Uniqueness claims"

HW: Suppose $S \subseteq V$ is linearly-independent

Then any vector $w \in$ Span(S)

has a unique expression.

Put the 2 notions together and get...

Def: A <u>basis</u> of an $\mathbb{F}$-vs V

is a set $\mathcal{B} \subseteq V$ (\mathcal{ })

such that

1) $V = \operatorname{span}(\mathcal{B})$

2) $\mathcal{B}$ is linearly independent

ex) i) $V = \mathbb{F}^n$ then "standard basis"

ex) $V = \mathbb{R}^3$
$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

is $e_1, e_2, \ldots, e_n$ $\left(e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \right)$

ii) $S = \{x_1 \ldots x_n\}$. $V = Fct(S, \mathbb{F})$

then the "Kronecker-delta" basis

$\beta = (\delta_1 \ldots \delta_n)$ defined as

$\delta_i : S \to \mathbb{F}$ $\quad \delta_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Try to prove this! (take $g \in V$ try $g = c_1 \delta_1 + \ldots + c_n \delta_n$)

iii) $V = \mathbb{F}[t]_{\leq n}$ has "standard basis"

$$\mathcal{B} = (1, t, t^2, t^3, \dots, t^n)$$

iv) $M_{2\times 2}(\mathbb{F})$ has "standard basis" $(m_1, m_2, m_3, m_4) = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$

<u>Prop</u>: Let $\mathcal{B}$ be set in V.

Then $\mathcal{B}$ is a <u>basis</u>

$\iff$ every $w \in V$ <u>can be expressed</u> !

<u>as a linear-combo of vectors in $\mathcal{B}$</u>

How to get **Basis** ?

Lemma: Let $S = (v_1 \dots v_n)$ subset, $\tilde{w} \in V$.

Let $\tilde{S} = (v_1 \dots v_n, w)$ Then

i) $\text{Span}(S) = \text{Span}(\tilde{S}) \iff w \in \text{Span}(S)$

ii) If S is LI, then so is $\tilde{S} \iff w \notin \text{Span}(S)$

Pf) (i) Assume $\text{Span}(S) = \text{Span}(\tilde{S})$.

Note $w \in \text{Span}(\tilde{S}) = \text{Span}(S)$ ✓

Now $w = c_1 v_1 + \dots + c_n v_n$ for $c_i \in \mathbb{F}$ (ie $w \in \text{Span}(S)$)

Take $\gamma \in \operatorname{Span}(\tilde{S})$. Then $\gamma = d_1 v_1 + \cdots + d_n v_n + d w$

Now plug in expression for $w \Rightarrow \gamma = d_1 v_1 + \cdots + d_n v_n + d(c_1 v_1 + \cdots + c_n v_n)$

$\Rightarrow \gamma = (d_1 + d c_1) v_1 + \cdots + (d_n + d c_n) v_n \in \operatorname{Span}(S)$ $\boxed{:)}$

Since $S \subseteq \tilde{S}$ $\operatorname{Span}(S) \subseteq \operatorname{Span}(\tilde{S})$

(ii) Assume, $\tilde{S}$ is also LI. If $w \in \operatorname{Span}(S)$

then $w = c_1 v_1 + \cdots + c_n v_n$. with not all $c_i = 0$

$(-1)v = (-v)$

comes from

$0_{\mathbb{F}} v = 0_V$

then $0_V = c_1 v_1 + \cdots + c_n v_n + (-1) w$

yet $\tilde{S}$ is LI $\rightarrow \leftarrow$

Now assume $w \notin \operatorname{span}(S)$, but that $\tilde{S}$ not LI

Then $\exists c_1 \ldots c_n, d \in \mathbb{F}$ not all $0_{\mathbb{F}}$ such that

$0_V = \underline{c_1 v_1 + \cdots + c_n v_n} + d w$. If $d = 0$ then this would contradict

S be LI. $\Rightarrow w = -\frac{c_1}{d} v_1 - \frac{c_2}{d} v_2 - \frac{c_3}{d} v_3 - \cdots - \frac{c_n}{d} v_n \rightarrow\leftarrow$

This will help us construct basis

Def: Say V is finite-dimensional if there is a finite subset S that spans V (ie, $V = \text{span}(S)$)

ex) i) $\mathbb{F}^n$

ii) $M_{m \times n}(\mathbb{F})$

iii) $\mathbb{F}[t]_{\leq n}$

iv) S finite set, $\text{Fct}(S, \mathbb{F})$

HW: v) Show that $V = Fct(\mathbb{Z}, \mathbb{F})$ not finite-dimensional)

vi) $\mathbb{F}[t]$ not finite dimensional

Prop: Let $S = (v_1 \dots v_n)$ be set that spans V.

a) Given L a linearly-independent subset of V, we obtain a basis for V by adjoining elements of S to L

b) Obtain a basis for V by

excluding elements in S. (if needed)

Pf) b) If S is LI nothing to do ✓ Assume S not LI

$\exists v_i \in \text{span}(v_1, \ldots v_{i-1}, v_{i+1}, \ldots, v_n)$

Call $\tilde{S} = (v_1 \ldots v_{i-1}, v_{i+1}, \ldots, v_n)$

Note $\text{Span}(\tilde{S}) = \text{span}(S)$ by last lemma.

If $\tilde{S}$ is now LI we're done. If not, repeat.

Eventually this must terminate. $\square$

"spans" = $\text{Span}(L) = V$

a) If L spans ✓ nothing to do.

If L doesn't span then $\exists v_i \in S$ such that

$v_i \notin \text{Span}(L)$. Because, if $S \subseteq \text{span}(L)$ then

$\text{Span}(S) \subseteq \text{span}(L)$ but $\text{span}(S) = V \rightarrow \Leftarrow$

Consider $\tilde{L} = (L, v_i)$ this remains LI by lemma above. Now mimick the above to finish the proof.

Cor: Every fd vector space has a basis.

Rmks: true for general VS, harder to prove. Uses "Zorns Lemma"

Towards Dimension

Prop: S, L finite subsets in V.

Assume i) S spans V

ii) L is linearly-ind

Then $|S| \geq |L|$

Pf) Take $S = (v_1, \dots, v_n)$ spanning. Take $L = (w_1, \dots, w_m)$ LI. Now since S spans, adjoining any vector makes the new list LD. Adjoin w_1 from L to get the list

$$(w_1, v_1, \dots, v_n).$$

Consider the following lemma.

If $z_1, \dots, z_k$ are Linearly dependent then $\exists j \in \{1, \dots, k\}$ such that

$$z_j \in \operatorname{span}(z_1, \dots, z_{j-1}) \quad \text{(notice the indices)}$$

Pf) Since $z_1, \dots, z_k$ are LD $\exists\, a_1, \dots, a_k \in \mathbb{F}$ not all zero, such that

$a_1 z_1 + \cdots + a_k z_k = 0_V$.

Let j be the largest index in $\{1, \ldots, k\}$ such that $a_j \neq 0$.

Then $v_j = -\frac{a_1}{a_j} v_1 - \frac{a_2}{a_j} v_2 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$:)

Back to the proof

- Since $(w_1, v_1, \ldots, v_n)$ are LD we can remove one of the v_i and still span
- Now repeat and adjoin w_2 to get ($\wedge$ = remove)

  $(w_1, w_2, v_1, \ldots, \hat{v}_i, \ldots, v_n)$

By the lemma above one of these vectors

must be in span of the previous vectors.

Since w_1, w_2 are part of LI list we know $w_2 \notin \text{span}(w_1)$, So $\exists v_j$ $j \in \{1 \ldots \hat{i}, \ldots, n\}$ st $v_j \in \text{span}(w_1, w_2, v_1, \ldots, v_{j-1})$ by previous lemma. Again remove that vector.

Continue for each step. At step k we have a LD list

$(w_1 \ldots w_k$, some v's with k of them removed$)$

Keep going and at each step the lemma above implies the list is LD, so that there is some v to remove.

This means there are at least as many v's as there are w's $\boxed{\because}$ (ugly proof $\because$)

Cor: V finite-dim VS and $\mathcal{B}$ a basis. Then

Exercise

a) Any other basis $\mathcal{B}'$ has the same # of vectors as $\mathcal{B}$

b) If S is finite subset spanning V then $|S| \geq |\mathcal{B}|$

c) If L is finite LI set then $|\mathcal{B}| \geq |L|$

$\Rightarrow$ Def: The dimension of a finite dim VS

V is defined to be the # of vectors in a basis

Next time: Linear transformations